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**LETTER TO THE EDITOR**

# Minkowskian instantons

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**Abstract.** I present solutions to a scalar and an SO(3, 1) gauge Minkowskian field theory, which are closely related to the Euclidean instanton solutions and are integrable. They have vanishing energy-momentum tensors and finite actions.

Recently, interest has been paid to the so called instanton solutions (Belavin *et al* 1975, 't Hooft 1976) of classical field theory. These are non-singular and integrable in an Euclidean space. They have been given some physical significance (Jackiw 1977) in the corresponding quantum theory.

My purpose, in this Letter, is to point out that closely related solutions to some Minkowskian field theories are also integrable.

I consider first a real scalar field with the following Minkowskian Lagrangian density:

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 + \lambda\phi^4. \tag{1}$$

The coupling constant $\lambda$ is real and the sign chosen is *opposite* to that case in which Euclidean instantons are found (Fubini 1976). The field equations

$$\partial^2\phi - 4\lambda\phi^3 = 0 \tag{2}$$

admit the solution

$$\phi = \left(\frac{2}{\lambda}\right)^{1/2} \frac{b}{x^2 - b^2 + \mathrm{i}\epsilon} \tag{3}$$

for an arbitrary real constant $b$.

The singularity of the solution (3) lies on a timelike surface and the $\mathrm{i}\epsilon$ is prescribed to avoid this singularity and make (3) integrable.

The Minkowskian action corresponding to the Lagrangian density (1) and the solution (3) is easily evaluated:

$$I = \int \mathrm{d}^4x\mathscr{L} = -\tfrac{2}{3}\mathrm{i}\pi^2/\lambda. \tag{4}$$

The presence of the imaginary unit in (4) lends to our solution the possibility of a tunnelling interpretation (Jackiw 1977) in the corresponding quantum theory.

Now I consider an SO(3, 1) Minkowskian gauge field, described by the Lagrangian density

$$\mathscr{L} = \tfrac{1}{4}F^{ab}_{\mu\nu}F^{\mu\nu}_{ab} - F^{\mu\nu}_{ab}(\partial_\mu A^{ab}_\nu + gA^{a}_{\mu c}A^{bc}_\nu) \tag{5}$$

where $F^{ab}_{\mu\nu}$ is antisymmetric in both types of indices and ($a, b = 0, 1, 2, 3$; $\mu, \nu = 0, 1, 2, 3$).

The equations of motion

$$F^{ab}_{\mu\nu} = \partial_\mu A^{ab}_\nu + gA^{\ a}_{\mu c}A^{bc}_\nu - (\mu \leftrightarrow \nu)$$
$$\partial^\mu F^{ab}_{\mu\nu} + gA^{\mu a}{}_c F^{bc}_{\mu\nu} - gA^{\mu b}{}_c F^{ac}_{\mu\nu} = 0 \tag{6}$$

admit the solution

$$A^{ab}_\mu = \frac{2}{g}\frac{\eta^a_\mu x^b - \eta^b_\mu x^a}{x^2 - b^2 + \mathrm{i}\epsilon}$$
$$F^{ab}_{\mu\nu} = \frac{4}{g}\frac{b^2}{(x^2 - b^2 + \mathrm{i}\epsilon)^2}(\eta^a_\mu\eta^b_\nu - \eta^b_\mu\eta^a_\nu). \tag{7}$$

The Minkowskian action corresponding to the Lagrangian density (5) and the solution (7) is

$$I = -16\mathrm{i}\pi^2/g^2. \tag{8}$$

The Minkowskian energy–momentum tensor for the SO(3, 1) gauge field and the improved energy–momentum tensor for the scalar field defined by

$$\theta^A_{\mu\nu} = -F^{ab}_{\mu\lambda}F_{\nu ab}^{\ \ \lambda} + \tfrac{1}{4}\eta_{\mu\nu}F^{ab}_{\lambda\rho}F^{\lambda\rho}_{ab} \tag{9}$$

$$\theta^\phi_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}[\tfrac{1}{2}(\partial_\lambda\phi)^2 + \lambda\phi^4] - \tfrac{1}{6}(\partial_\mu\partial_\nu\phi^2 - \eta_{\mu\nu}\partial^2\phi^2) \tag{10}$$

both vanish for the solutions (3) and (7).

It may be useful at this point to give the following remark. The self-duality properties of the SU(2) Euclidean instantons have been helpful criteria for their construction and are responsible for the vanishing of the Euclidean energy–momentum tensor. In the above Minkowskian theories, however, the vanishing of the energy–momentum tensor seems to be the essential property†. It may serve therefore to find other Minkowskian solutions.

Finally, it may be worth noting that the following invariant of the SO(3, 1) gauge field defined by

$$Q_1 = \int \mathrm{d}^4x \tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}F^{ab}_{\mu\nu}F_{\lambda\rho ab} \tag{11}$$

vanishes for the solution (7). However, the invariant

$$Q_2 = \int \mathrm{d}^4x \tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}F^{ab}_{\mu\nu}F^{cd}_{\lambda\rho}\epsilon_{abcd} = 128\mathrm{i}\pi^2/g^2. \tag{12}$$

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† I am indebted to Professor L O'Raifeartaigh for this remark.